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Neutron matter with a model interaction

M.Ya. Amusia[1,2] and V.R. Shaginyan[3,4,a]

[1] The Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel
[2] A.F. Ioffe Physical-Technical Institute, 194021 St. Petersburg, Russia
[3] Petersburg Institute of Nuclear Physics, 188350 Gatchina, Russia
[4] Department of Physics, University of Washington, Seattle, WA 98195, USA

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Abstract. An infinite system of neutrons interacting by a model pair potential is considered. We investigate a case when this potential is sufficiently strong attractive, so that its scattering length a tends to infinity, $a \to -\infty$. It appeared, that if the structure of the potential is simple enough, including no finite parameters, reliable evidences can be presented that such a system is completely unstable at any finite density. The incompressibility as a function of the density is negative, reaching zero value when the density tends to zero. If the potential contains a sufficiently strong repulsive core then the system possesses an equilibrium density. The main features of a theory describing such systems are considered.

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There exists a well known problem in many body physics related to the description of the ground state properties of an infinite system composed of interacting fermions. In general, this description is based usually on tedious numerical calculations, particularly when the interaction is rather strong. The well known exceptions from this situation, when it is possible to calculate the ground state properties analytically, are the Random Phase Approximation (RPA) for a high density electron gas [1] and the low density approximation for dilute gases [2]. In both cases the kinetic energy T_k is much bigger then the interaction energy E_{int} of the system. This allows to apply some kind of a perturbation theory. In the case of homogeneous electron liquid it turns out that the analytical RPA-like description is also possible not only at very high but medium densities when $T_k \sim E_{int}$ [3,4]. Similar extension of the range of validity is impossible in the case of fermion systems at low densities ρ: there the gas approximation is not applicable if $T_k \sim E_{int}$. In the cases when the pair interaction is attractive and sufficiently strong, the system can have a quasi equilibrium or equilibrium states in which $T_k \simeq E_{int}$. On the other hand, these states are preceded by special points with density ρ values at which the incompressibility $K(\rho)$ of the system tends to zero. Thus, if it would be possible to predict the existence of such points then in principle it would become possible to conclude that the system has at least a quasi equilibrium state.

In this Short note we address the above mentioned problem and consider the ground state properties of the infinitely extended multi-fermion system. We demonstrate that it can be done analytically provided that the pair interaction between fermions is characterized only by the scattering length $a \to -\infty$. One can say in this case that the scattering length is the dominant parameter of the problem under consideration. Such an investigation is of great importance since it can be applied to fermion systems interacting via potentials with not only infinite, but also sufficiently large a. For instance, the scattering length a of neutron-neutron interaction is about -20 fm, thus greatly surpassing the radius of the interaction r_0. On the other hand, it is possible now to prepare artificially a system composed of Fermi atoms interacting by an artificially constructed potential with almost any desirable scattering length, similarly to that how it is done for the trapped Bose gases, see e.g. [5]. An experimental study, performed on such Fermi-system would be of great importance presenting new information on the behavior of dilute gases and the gas-liquid phase transition.

Let us consider the interaction of two isolated particles. We assume that this interaction is of finite radius r_0, which is small, so that $p_F r_0 \ll 1$ ($p_F = (3\pi^2\rho)^{1/3}$ is the Fermi momentum), but its strength is such that the scattering length is negative and infinitely big, $a \to -\infty$. We assume also, that the density ρ of the system under consideration is homogeneous. As it will be demonstrated below, in such a case the system is located in the vicinity of a phase transition, transforming it into a strongly correlated one. Therefore, the problem of calculating its ground state properties has to be treated for the most part qualitatively.

[a] e-mail: vrshag@thd.pnpi.spb.ru

Let us start considering general properties of a Fermi system with some attractive two-particle bare interaction $V(r)$, which is sufficiently weak to create a two-particle bound state. Assume, that the scattering length a corresponding to this potential is negative and finite. The ground state energy density $E(\rho)$ can be approximated by Skyrme-like expression [6],

$$E(\rho) = \frac{3p_F^2}{10M}\rho + t_0\rho^2 + t_3\rho^{7/3}, \tag{1}$$

Here M is the particle mass, and ρ is their density in the system. The first term of (1) is the kinetic energy T_k, while the second and the third are related to the interaction energy E_{int} determined by the potential $V(r)$. The second term which is proportional to t_0 gives a proper description in the gas limit. The third term provides the behavior of $E(\rho)$ at higher densities, including that of the equilibrium density. Such structure of $E(\rho)$ appears if the interaction is sufficiently attractive so that $t_0 < 0$, and $t_3 > 0$. Note, that (1) presents at least a qualitative description of the system under consideration giving a rather simple and reasonable presentation of the function $E(\rho)$. A more precise picture of the energy dependence upon density can be obtained using more sophisticated functionals for the ground state energy [7].

If the potential V is of short range and purely attractive, then in the Hartree-Fock approximation the ground state energy E_{HF} is given by the following expression

$$E_{HF}(\rho) = \frac{3p_F^2}{10M}\rho + t_{HF}\rho^2, \tag{2}$$

where the parameter $t_0 = t_{HF}$, being negative, is entirely determined by the potential $V(r)$. For instance, in the case of a short range δ-type interaction one has $t_{HF} = -v_0/4$, with v_0 being the corresponding strength of the potential. Equation (2) shows that at small densities $E_{HF} > 0$ due to the kinetic energy term, but at sufficiently high densities $\rho \to \infty$ the Hartree-Fock energy becomes dominating, leading to the collapse of the system, with $E_{HF} \to -\infty$. Keeping in mind that the Hartree-Fock approximation gives the upper limit to the binding energy $E_{HF} \geq E$, one can conclude that the system does not have, in this case, an equilibrium density ρ_e and energy E_e since $E_e \to -\infty$ when $\rho \to \infty$ [8]. Note, that for a given and finite total number of particles N, the HF energy is not going to infinity and the system collapses into a small volume with the radius r_0, with the density $\rho \sim N/r_0^3$. It is evident that the function $E(\rho)$ is positive at small densities, if the parameter t_0 is finite. Therefore, it must have at least one maximum at the density ρ_m before it becomes negative, on the way to $E \to -\infty$. If the potential $V(r)$ includes a repulsive core at sufficiently short distances, then $t_3 > 0$. As a result, the system has an equilibrium density and energy, ρ_e and E_e, respectively, determined by the repulsive core strength and its radius $r_c \sim r_0$.

Now let us apply (1) to demonstrate the most important features of the system under consideration:

a) when $\rho \to 0$ the third term on r.h.s. in (1) can be omitted. The kinetic energy is relatively very big, $T_k \gg E_{int}$, and $t_0 \sim a$, with $a < 0$ being the scattering length. In that case we have a dilute Fermi gas with positive pressure P and incompressibility K, the latter being determined by the equation, see e.g. [9],

$$K(\rho) = \rho^2 \frac{dE^2(\rho)}{d\rho^2}. \tag{3}$$

b) on the way to higher densities, which can be achieved by applying an external pressure, the system reaches the density $\rho_{c1} < \rho_m$ at which the incompressibility is equal to zero, $K(\rho_{c1}) = 0$. Remembering that at the maximum the second derivative is negative, one can conclude, as it is seen from (3), that $K(\rho_m) < 0$. In the range $\rho_{c2} \geq \rho \geq \rho_{c1}$ the incompressibility is negative, $K < 0$, and as a result the system becomes totally unstable. In fact, in this density range all calculations of the ground state energy are meaningless since such a system cannot exist and thus be observed experimentally [10];

c) at some point $\rho = \rho_{c2} > \rho_m$ the contribution due to the repulsive core becomes sufficiently strong to prevent the further collapse of the system. The incompressibility attains $K = 0$ at $\rho_{c2} < \rho_e$, being positive at the higher densities. Finally, the system becomes stable at $\rho > \rho_{c2}$, reaching equilibrium density at ρ_e with equilibrium energy equal to E_e. It is obvious that $K(\rho_e) > 0$ being proportional to the second derivative at the minimum, see (3). It should be kept in mind that in this density domain, $\rho \geq \rho_{c2}$, the function $E(\rho)$ is determined by the repulsive part of the potential which makes $t_3 > 0$. As it was mentioned above, without this component of $V(r)$ the system's energy would infinitely increase, $E_{HF} \to -\infty$, with density growth, $\rho \to \infty$, thus inevitably collapsing.

One could expect in principle the existence of metastable states at $\rho > \rho_{c1}$ if the potential $V(r)$, even being pure attractive, would have a complicated structure. It can be said that there could exist parameters of $V(r)$, which are able to open the possibility for the metastable states to be formed. On the other hand, a system of fermions interacting via a short-range, finite scattering-length, δ-type potential V_δ , can be stable only in the dilute gas regime. While at the densities $\rho \geq \rho_{c1}$ the incompressibility K becomes negative, the system collapses. Indeed, the potential V_δ has no structure to ensure any metastable states at the densities $\rho \geq \rho_{c1}$. As a result, one can write down a dimensionless expression for the ground state energy as a function of the only variable $z = p_F a$ [2,11],

$$\alpha E(z) = z^5(1 + \beta(z)), \tag{4}$$

with $\alpha = 10\pi^2 M a^5$. In the low density limit, $|ap_F| \ll 1$ and when the interaction has the radius r_0, (4) reads [11],

$$\alpha E(z) = z^5\left[1 + \frac{10}{9\pi}z + \frac{4}{21\pi^2}(11 - 2\ln 2)z^2 + \left(\frac{r_0}{a}\right)^3 z^3 \gamma\left(\frac{r_0}{a}, z\right) + \ldots\right]. \tag{5}$$

Here the function $\gamma(y,z)$ is of the order of one, $\gamma(y,z) \sim 1$. It is seen from (5) that as soon as the scattering length becomes big enough, $|a| \gg r_0$, one can omit the contribution coming from the function γ and neglect all the term proportional to $(r_0/a)^3$. Then (5) reduces to (4). Thus, in the case $|a| \to \infty$ we can use (4) to determine the ground state energy E. Equation (4) is valid up to the density ρ_{c1} which is a singular point of the function $\beta(z)$, since beyond this point $K < 0$, and the system is completely unstable. On the other hand, there is no physical reasons to have another irregular point in the region $0 \le \rho \le \rho_{c1}$. We note, that (5) is as well valid up to its own density $\rho'_{c1} \simeq \rho_{c1}$, provided $|a| \gg r_0$. Using (3) for the incompressibility, one can calculate the position of the point z_{c1} where $K = 0$. Denoting the corresponding z as $z_{c1} = c_0$, where c_0 is a dimensionless number, one is led to the conclusion that $\rho_{c1} \sim |a|^{-3}$ provided a is sufficiently large to be the only dominating parameter. The system has only one stable region at small densities $\rho \le \rho_{c1}$ which decreases and even vanishes as soon as $a \to -\infty$. One could expect that $|c_0| \to \infty$ as soon as a becomes the dominant parameter so that the above given expression for $\beta(z)$ is valid in the whole domain $|z| \le \infty$. On the other hand, there exists another singular point z_{c2} in the function $E(\rho)$ and the position of this point which corresponds to ρ_{c2} depends mainly on the parameters such as r_0, core radius r_c of $V(r)$ rather than on the scattering length a. As to the function $\beta(z)$, by definition it does not contain any information about ρ_{c2}. Therefore, in order to take into account the existence of ρ_{c2}, one has to recognize that c_0 is finite, and the densities ρ_{c1} and ρ_{c2} are different. As a result, the function β is determined in fact only in the region $|z| \le |z_{c1}|$.

Now let us consider the calculation of the ground state energy E of the system when the density approaches $\rho \to \rho_{c1} \sim |a|^{-3}$ from the low density side. As a rule, points ρ_{c1} and ρ_{c2} are missed in calculations because of the lack of the self consistency [12–14], which relates the linear response function of system with its incompressibility K,

$$\chi(q \to 0, i\omega \to 0) = -\left(\frac{d^2E}{d\rho^2}\right)^{-1}. \tag{6}$$

As we shall see below, these points can give important contributions to the ground state energy. To see it we express the energy of a system in the following form (see e.g. [9]),

$$E(\rho) = T_k + E_H - \frac{1}{2}\int \left[\chi(q,i\omega,g) + 2\pi\rho\delta(\omega)\right] v(q) \frac{d\mathbf{q}\, d\omega\, dg}{g(2\pi)^4}, \tag{7}$$

where $\chi(q,i\omega,g)$ is the linear response function on the imaginary axis and $v(q) = gV(q)$, with $V(q)$ being the Fourier image of $V(r)$. The integration over ω goes from $-\infty$ to $+\infty$, while the integration over the coupling constant g runs from zero to the real value of the coupling constant, i.e. to $g = 1$. At the point $\rho = \rho_{c1}$ the linear response function has a pole at the origin of coordinates $q = 0$, $\omega = 0$ due to (6). At the densities $\rho > \rho_{c1}$ the function $\chi(q,i\omega)$ has poles at finite values of the momentum q and frequencies $i\omega$. This prevents the integration over $i\omega$, making the integral in (7) divergent. Thus, we conclude that it is the contribution of these poles that reflects the system's instability in the density range $\rho_{c1} \le \rho \le \rho_{c2}$. Note, that violations of (6) lead to serious errors in the calculation of the ground state energy. Equation (7) can be rewritten, explicitly accounting for the effective interparticle interaction $R(q,i\omega,g)$, (see e.g. [14]), in the following form

$$E(\rho) = T_k + E_H - \frac{1}{2}\int \left[\frac{\chi_0(q,i\omega)}{1 - R(q,i\omega,g)\chi_0(q,i\omega)} + 2\pi\rho\delta(\omega)\right] v(q) \cdot \frac{d\mathbf{q}\, d\omega\, dg}{g(2\pi)^4}. \tag{8}$$

Here χ_0 is the linear response function of noninteracting particles, while χ is given by the following equation

$$\chi(q,\omega) = \frac{\chi_0(q,\omega)}{1 - R(q,\omega)\chi_0(q,\omega)}. \tag{9}$$

It is seen from (6) and (9) that the denominator $(1 - R\chi_0)$ vanishes at $\rho \to \rho_{c1}$ while the radius of correlation tends to infinity [10]. Thus, it is impossible to present the denominator as a power series in $R\chi_0$ approximating the expansion by the finite number of terms. This result is quite obvious since ρ_{c1} is a singular point in the function $E(\rho)$ which makes it impossible to expand that function in the vicinity of this point. Therefore, one should try to satisfy (6) in order to get proper results for the ground state calculations in the vicinity of the instability points. Such an approach was suggested in [3,4,14] and is based on the exact functional equation for the effective interaction $R(q,\omega,g)$,

$$R(q,\omega,g_0) = g_0 v(q) - \frac{1}{2}\frac{\delta^2}{\delta\rho^2(q,\omega)} \cdot \int \frac{\chi_0(k,iw)}{1 - R(k,iw,g)\chi_0(k,iw)} v(q) \cdot \frac{d\mathbf{k}\, dw\, dg}{g(2\pi)^4}. \tag{10}$$

As a result, the linear response function χ given by (9) automatically satisfies (6) [14]. Our preliminary calculations [14,15], based on (9) and applied to the case when the scattering length is sufficiently large but finite, confirm the result that $\rho_{c1} \sim |a|^{-3}$.

Let us suppose for a while that the bare potential is pure attractive. Then, the interval of the densities $[0, \rho_{c1}]$ within which the system is stable vanishes with the growth of $|a|$. As a result, in the limit $a = -\infty$ the incompressibility becomes negative $K \le 0$, making the considered system completely unstable at any density. Thus, the point at which $a = -\infty$ is the only point of the system's instability at all the densities. As soon as the scattering length deviates from its infinite value, that is $+\infty > a > -\infty$

the system comes back to its stable state at list in the range of the density values $\rho < \rho_{c1} \sim |a|^{-3}$. It is of interest to understand whether it is possible to prove by e.g. numerical calculations, that $\rho_{c1} \sim |a|^{-3}$ when $a \to -\infty$. From our point of view, at least at this moment, the answer is "no". We are dealing with a system located in the vicinity of a phase transition, which transforms it into a strongly correlated one. As a result, it is hard to believe that the numerical calculations could be reliable. On the other hand, it is not really necessary to carry out numerical calculations if the problem allows a qualitative analysis. It was argued above, that there exists the only parameter to characterize the system which is the scattering length a. In fact the scattering length determines only the specific point ρ_{c1} at which the incompressibility vanishes, separating the region of a dilute gas from the region of the system's instability. As soon as $a \to -\infty$ this last and the only parameters vanishes, driving the point $\rho_{c1} \sim |a|^{-3}$ of the curve $E(\rho)$ to the origin of coordinates. Thereafter, the system becomes unstable at all densities. And vice versa, as soon as the scattering length becomes finite the system is stable at list within the interval $\rho \leq \rho_{c1} \sim |a|^{-3}$.

Note, that as it follows from our consideration, any Fermi system possesses an equilibrium density and energy if the bare particle-particle interaction contains a repulsive core and its attractive part is strong enough, so that $a \to -\infty$. Indeed, at sufficiently small densities the ground state energy is negative (since the incompressibility $K \leq 0$) and the system will collapse until the core stops the density growth. Therefore, the minimal value of the ground state energy must be negative when the repulsive core will enter the play to prevent the system from the further collapse. It is worth to remark, that superfluid correlations cannot stop the system squeezing, since their contribution to the ground state energy being negative increases in the absolute value with the growth of the density.

A liquid similar to the model one considered in this paper exists in Nature. This is liquid ^3He. If a helium dimer exists, its bound energy does not exceed 10^{-4} meV while the ground state energy of helium liquid is about $2 * 10^{-1}$ meV per atom [16]. Because of this huge difference in binding energies, it is evident that there is no essential contribution coming from the binding energy of the dimer to the ground state energy of the liquid. In fact, the numerical calculations show that the pair potential is rather weak to produce the dimer He_2 [16]. Thus, one can reliably consider an infinite homogeneous system of Helium atoms as consisting of particles interacting via pair potential, characterized by a very big but finite scattering length $|a| \gg r_0$. Let us make also the following additional remark. It seems quite probable that the neutron-neutron scattering length ($a \simeq -20$ fm) is sufficiently large to permit the neutron matter to have an equilibrium energy and density [15]. Therefore, calculations of a neutron matter satisfying (7) are quite desirable.

In summary, the homogeneous system of interacting fermions was considered. It was shown that when the scattering length a is negative and sufficiently large the fermion matter becomes a strongly correlated system at the densities $\rho \sim |a|^{-3}$. Therefore, the consideration of such a system is connected to a number of problems which yet persist and have to be resolved. At the same time, the qualitative consideration presented above gives strong evidences that the point ρ_{c1} at which the incompressibility vanishes is defined by $\rho_{c1} \sim |a|^{-3}$ provided the scattering length is the dominant parameter of the problem. Thus, a homogeneous system composed of fermions, interacting via a pure attractive potential, at $a \to -\infty$ is completely unstable at all the densities, with the incompressibility as a function of the density being always negative. As soon as the density ρ goes to zero the incompressibility goes to zero as well.

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